

AN APPLICATION OF JACK'S LEMMA FOR THE MINIMUM POINT

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ABSTRACT. For the analytic function $f(z)$, H. Shiraishi and S. Owa (Stud. Univ. Babeş-Bolyai Math. **55**(2010), 207-211) have shown a theorem for the minimum value of $|f(z)|$. In this paper, we discuss an application of this theorem and some corollaries.

1. INTRODUCTION

Let the set of \mathbb{U} be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{H}[a_0, n]$ denote the class of functions $p(z)$ of the form

$$p(z) = a_0 + \sum_{k=n}^{\infty} a_k z^k$$

which are analytic in \mathbb{U} for some $a_0 \in \mathbb{C}$ and a positive integer n .

The basic tool in the proof of our results is the following lemma due to H. Shiraishi and S. Owa [4].

Lemma 1. *Let $p(z) \in \mathcal{H}[a_0, n]$ with $p(z) \neq 0$ for all $z \in \mathbb{U}$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$\min_{|z| \leq |z_0|} |p(z)| = |p(z_0)|,$$

then

$$\frac{z_0 p'(z_0)}{p(z_0)} = -m$$

and

$$\operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 \geq -m,$$

where

$$m \geq n \frac{|a_0 - p(z_0)|^2}{|a_0|^2 - |p(z_0)|^2} \geq n \frac{|a_0| - |p(z_0)|}{|a_0| + |p(z_0)|}.$$

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2. MAIN THEOREM

Applying Lemma 1, we derive

Theorem 1. *Let the function $f(z)$ given by*

$$f(z) = a_n z^n + a_{n+l} z^{n+l} + a_{n+l+1} z^{n+l+1} + \dots \quad (a_n, a_{n+l} \neq 0)$$

be analytic in \mathbb{U} and $f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. If there exists a point $z_0 \in \mathbb{U} \setminus \{0\}$ such that

$$\min_{|z| \leq |z_0|} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = n - m \tag{1}$$

and

$$(n - m) \left(\operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \right) \leq (n - m)^2, \tag{2}$$

where

$$m \geq l \frac{|a_n z_0^n - f(z_0)|^2}{|a_n z_0^n|^2 - |f(z_0)|^2} \geq l \frac{|a_n z_0^n| - |f(z_0)|}{|a_n z_0^n| + |f(z_0)|}.$$

Proof. We define the function $p(z)$ by

$$\begin{aligned} p(z) &= \frac{f(z)}{z^n} \\ &= a_n + a_l z^l + a_{l+1} z^{l+1} + \dots \end{aligned}$$

Then, $p(z) \in \mathcal{H}[a_n, l]$ and $p(0) = a_n \neq 0$. Furthermore, by the assumption of the theorem, $|p(z)|$ takes its minimum value at $z = z_0$ in the closed disk $|z| \leq |z_0|$. It follows from this that

$$|p(z_0)| = \frac{|f(z_0)|}{|z_0|^n} = \frac{\min_{|z| \leq |z_0|} |f(z)|}{|z_0|^n} = \min_{|z| \leq |z_0|} |p(z)|.$$

Therefore, applying Lemma 1 to $p(z)$, we observe that

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{z_0 f'(z_0)}{f(z_0)} - n = -m$$

which shows (1) and

$$\begin{aligned} \operatorname{Re} \frac{z_0 p''(z_0)}{p'(z_0)} + 1 &= \operatorname{Re} \left(-n - 1 + \frac{\frac{z_0 f''(z_0)}{f'(z_0)} + 1 - n}{1 - n \frac{f(z_0)}{z_0 f'(z_0)}} \right) + 1 \\ &= -\frac{n - m}{m} \left(\operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 - n \right) - n \\ &\geq -m \end{aligned}$$

which implies (2), where

$$m \geq l \frac{|a_n - p(z_0)|^2}{|a_n|^2 - |p(z_0)|^2} = l \frac{|a_n z_0^n - f(z_0)|^2}{|a_n z_0^n|^2 - |f(z_0)|^2} \geq l \frac{|a_n z_0^n| - |f(z_0)|}{|a_n z_0^n| + |f(z_0)|}.$$

This completes the assertion of Theorem 1. \square

Letting $l = n$ in Theorem 1, we obtain

Corollary 1. *Let the function $f(z)$ given by*

$$f(z) = a_n z^n + a_{2n} z^{2n} + a_{2n+1} z^{2n+1} + \dots \quad (a_n, a_{2n} \neq 0)$$

be analytic in \mathbb{U} and $f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. If there exists a point $z_0 \in \mathbb{U} \setminus \{0\}$ such that

$$\min_{|z| \leq |z_0|} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = n - m \leq 0$$

and

$$\operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq n - m,$$

where

$$m \geq n \frac{|a_n z_0^n - f(z_0)|^2}{|a_n z_0^n|^2 - |f(z_0)|^2} \geq n \frac{|a_n z_0^n| - |f(z_0)|}{|a_n z_0^n| + |f(z_0)|}.$$

Moreover, putting $n = 1$ and $a_n = 1$ in Theorem 1, we get the following corollary due to M. Nunokawa and S. Owa [3].

Corollary 2. *Let the function $f(z)$ given by*

$$f(z) = z + a_{l+1} z^{l+1} + a_{l+2} z^{l+2} + \dots \quad (a_{l+1} \neq 0)$$

be analytic in \mathbb{U} and $f(z) \neq 0$ for $z \in \mathbb{U} \setminus \{0\}$. If there exists a point $z_0 \in \mathbb{U} \setminus \{0\}$ such that

$$\min_{|z| \leq |z_0|} |f(z)| = |f(z_0)|,$$

then

$$\frac{z_0 f'(z_0)}{f(z_0)} = 1 - m \leq 0$$

and

$$\operatorname{Re} \frac{z_0 f''(z_0)}{f'(z_0)} + 1 \geq 1 - m,$$

where

$$m \geq l \frac{|z_0 - f(z_0)|^2}{|z_0|^2 - |f(z_0)|^2} \geq l \frac{|z_0| - |f(z_0)|}{|z_0| + |f(z_0)|}.$$

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